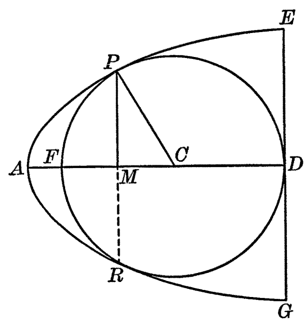
Let $PFRD$ be the given circle whose center C is the origin of rectangular coördinates. Then, without loss of generality, the parabola may be assumed to have its axis on the axis of abscissas AD.

Let EAG be the required parabola. Then, if r is the radius of the circle, we have $x^2 + y^2 = r^2$ for the circle, and $y^2 = 2p(x + k)$ for the parabola, where p is the distance from the focus to the directrix and k is the distance AC.

Since P is the point of tangency, solving these two simultaneous equations subject to the condition of tangency, we have $r^2 - 2kp + p^2 = 0$.



Hence, $k = p/2 + r^2/2p$.
Then

$$AD = AC + CD = \frac{p}{2} + \frac{r^2}{2p} + r = \frac{(r+p)^2}{2p},$$

and

$$ED = \sqrt{2p(AD)} = (r + p).$$

Area of $EAG = a = \frac{2}{3}AD \cdot EG = \frac{4}{3}(r + p)^2/2p \cdot (r + p) = \frac{2}{3}(r + p)^3/p$.
Equating to zero the derivative of a with respect p, we have

$$\frac{da}{dp} = \frac{2}{3} \cdot \frac{3p(r+p)^2 - (r+p)^3}{p^2} = 0;$$

whence

$$(p + r)^2 = 0 \quad \text{and} \quad 2p - r = 0.$$

Hence, $p = -r$ or $p = \frac{1}{2}r$. The value $p = -r$ gives neither a maximum nor a minimum. The value $p = \frac{1}{2}r$ gives a minimum, and the equation of the corresponding parabola is $y^2 = r(x + 5r)/4$.

A similar solution was received from the PROPOSER.

359. Proposed by W. D. CAIRNS, Oberlin College.

Examine for maxima and minima

$$f(x) = e^{-cx}(1 + \cos x) \quad (c > 0)$$

SOLUTION BY A. M. HARDING, University of Arkansas.

$$f'(x) = -e^{-cx}(c + c\cos x + \sin x).$$

$$f''(x) = e^{-cx}(c^2 + c^2\cos x + 2c\sin x - \cos x).$$

Now $f'(x) = 0$ only when $c + c\cos x + \sin x = 0$, that is, when

$$c(1 + \cos x) + \sin x = 0,$$

or

$$2c\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = 0.$$

Hence,

$$\cos\frac{x}{2} = 0, \quad \text{and} \quad x = \pi, 3\pi, 5\pi, \cdots (2n-1)\pi, \cdots$$

or

$$c\cos\frac{x}{2} + \sin\frac{x}{2} = 0, \quad \text{and} \quad x = 2\arctan(-c).$$

When $x = \pi, 3\pi, 5\pi, \cdots, (2n-1)\pi, \cdots, f''(x) = -c\,(2n-1)\pi$, which is always positive.

Hence, the minimum value of $f(x)$ is obtained by giving x any of these values. When $x = 2\arctan(-c)$, that is, $c\cos x/2 + \sin x/2 = 0$,

$$f''(x) = e^{-cx}[\{c^2(1 + \cos x) + c\sin x\} + c\sin x - \cos x]$$

$$= e^{-cx}\left[2c\cos\frac{x}{2}\left(c\cos\frac{x}{2} + \sin\frac{x}{2}\right) + 2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right]$$

$$= e^{-cx}\left[2c\sin\frac{x}{2}\cos\frac{x}{2} - \cos x\right] = e^{-cx}\left[2\sin\frac{x}{2}\left(-\sin\frac{x}{2}\right) - \cos x\right]$$

$$= -e^{-cx}\left[2\sin^2\frac{x}{2} + \cos x\right] = -e^{-cx}[1 - \cos x + \cos x] = -e^{-cx}.$$

Now $-e^{-cx}$ is negative for the above value of x. Hence the maximum value of $f(x)$ is obtained by giving x the value $2\arctan(-c)$.

Also solved by W. C. EELLS, PAUL CAPRON, H. C. FEEMSTER, G. W. HARTWELL and the PROPOSER.

MECHANICS.

286. Proposed by C. N. SCHMALL, New York City.

A slightly elastic string is just long enough to reach between two hooks on the same horizontal line. A ring of weight w is placed at its middle point. Show that the ring will sink through a distance $h = a\sqrt[3]{3e\omega/2}$, where e is the elasticity of the string and $2a$ the distance between the two hooks.

SOLUTION BY B. F. FINKEL, Drury College.

Since w is at the middle point of the string, the tension T in the two halves of the string is the same when the string is in equilibrium. Let θ be the angle which the string makes with the horizontal line.